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A resolution of the SU(3) outer multiplicity problem and computation of Wigner coefficients for SU(3)

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Abstract. A simple algorithm is given for the resolution of the SU(3) multiplicity problem and the computation of SU(3) Wigner coefficients using a complete set of $\mathcal{U}(2)\otimes \mathrm{U}(3)$ Bargmann tensors classified by operator patterns. Null space properties of these tensors are easily derived. Their structure is such that a direct one-to-one correspondence is shown to exist between the Bargmann tensors and the terms in the Clebsch-Gordan series for SU(3) as derived by O'Reilly. Finally, the resolution presented herein is shown to be concordant with an alternative resolution advocated long ago by Hecht.

1. Introduction

The computation of Wigner and Racah coefficients for the SU(n) groups is known to be fraught with difficulties for $n > 2$, which arise because the reduction of the Kronecker product of two irreducible representations of the group is generally not multiplicity free. This problem is usually referred to as the outer multiplicity problem. As a result, there is a degree of arbitrariness in the resolution of the orthogonality of the couplings. In physics parlance, one would say that there are missing labels which could unambigously identify individual members of a complete set of orthogonal coupled states. SU(n) invariant operators belonging to the su(n) enveloping Lie algebras have been proposed to supply the supplementary labels (see, e.g., Moshinsky 1963) but, beside being arbitrary to some extent, they are usually tensors of higher degree in the Lie algebras and therefore rather cumbersome to use.

It was realised by Biedenharn, Louck and collaborators (see the review article by Louck (1970)) that a clever use of the powerful Wigner-Eckart theorem could assist in the resolution of the SU(n) outer multiplicity problem. One has only to introduce a basic set of SU(n) tensors classified with the help of properties intrinsically related to the weight structure of the su(n) Lie algebras and their representations in order to supply the missing labels. Such a labelling scheme would be referred to as an *intrinsic* or *canonical* labelling. For the SU(n) groups, such a canonical labelling scheme is provided by the dual use of the Gel'fand basis labelling scheme by nested $\mathrm{U}(i) \supset \mathrm{U}(i-1)$ partitions. The corresponding pair of patterns are referred to as lower and upper (operator) patterns. The lower pattern enumerates the components of an irreducible tensor while the operator pattern describes in a compact fashion the structural properties of the tensor (shift properties, null space properties, etc) and consequently the formal structure of the corresponding Wigner coefficients.

In view of the success of the analysis of Biedenharn and collaborators in uncovering the formal structure of the SU(n) Wigner–Racah calculus in an illuminating way, one is encouraged to attempt an explicit realisation of their tensor operators. Such a concrete realisation would greatly facilitate the computation of Wigner and Racah coefficients for SU(n) (and more specifically of SU(3)) which are important in, for example, high energy and nuclear physics.

We partially unveiled such a practical framework in a recent paper on the structure of Bargmann tensors (Le Blanc and Rowe 1986). It was shown there that the use of an extended complementarity principle allowed the construction and classification by $\mathcal{U}(2)\otimes \mathrm{U}(3)$ operator patterns of a set of basic tensors defined in and acting on a Hilbert space of $\mathcal{U}(2)\otimes \mathrm{U}(3)$ Bargmann polynomials labelled by Young diagrams of at most two rows. It was shown there that the operator pattern acquires a group theoretical meaning since it refers to the tensorial properties of the U(3) tensors under transformations by the complementary $\mathcal{U}(2)$ group. We thus offered a partial but concrete resolution of the SU(3) outer multiplicity problem which has the interesting feature of providing a set of missing labels to the Wigner operators while retaining all the essential features of the Biedenharn *et al* analysis. We illustrated the effectiveness of our approach by giving analytical expressions for the isoscalar coefficients needed to perform the multiplicity-free coupling by $[\lambda 0]$ and $[0\mu]$ SU(3) tensors (Le Blanc and Rowe 1986).

The purpose of this paper is to illustrate how the corresponding $[\lambda 0]$ and $[0\mu]$ Bargmann tensors can be used to build a complete set of $[\lambda\mu]$ SU(3) tensors belonging to a given multiplicity set and how one can use them to give a concrete and unequivocal resolution to the SU(3) outer multiplicity. Towards this end, we will show that the tensors belonging to a given multiplicity set can be unambiguously put into one-to-one correspondence with each term in the Clebsch–Gordan series for SU(3) as derived by O'Reilly (1982) and how this correspondence allows easy identification of their corresponding null spaces.

An explicit algorithm for the calculation of the $\mathrm{SU}(3)\supset \mathrm{SU}(2)\times \mathrm{U}(1)$ Wigner coefficients in a Gel'fand basis is then fully expounded in § 6. The algorithm is based on an ordered Gramm–Schmidt process. It is simple enough to allow algebraic expressions to be obtained for low multiplicity cases and computer calculations in general. We note that the computation of matrix elements of the basic Bargmann tensors does not rely on a recursive process and is therefore very efficient. An example is given in § 7. Computation of $\mathrm{SU}(3)\supset \mathrm{SO}(3)$ Wigner coefficients is also briefly discussed.

It will be shown (at least for the important case of self-conjugate tensors) that the ordering given complies with a resolution of the outer multiplicity problem advocated long ago by Hecht (1965). According to Hecht a desirable resolution would be one having the property that the matrix formed by a restricted set of SU(3) Wigner coefficients, defining uniquely the more general coupling coefficients, naturally assumes a lower triangular form. We show that, for the self-conjugate tensors, the Wigner coefficients obtained with our ordering reproduce this property and we conjecture that this result is true in general. However, we find that our ordered tensors have null space inclusion properties different from those of Biedenharn's abstract set. A comparison is made in § 8.

2. Overview

A first step towards the resolution of the multiplicity problem for SU(3) is the derivation of a closed formula for its Clebsch-Gordan series:

$$[\lambda_1\mu_1]\otimes[\lambda_2\mu_2]=\sum_{\lambda_3\mu_3} l_{[\lambda_1\mu_1][\lambda_2\mu_2];[\lambda_3\mu_3]}[\lambda_3\mu_3] \tag{2.1}$$

where $l_{[\lambda_1\mu_1][\lambda_2\mu_2];[\lambda_3\mu_3]}$ (simply denoted l in the following) is the multiplicity of the unirrep $[\lambda_3\mu_3]$ in the Kronecker product of the unirreps $[\lambda_1\mu_1]$ and $[\lambda_2\mu_2]$. It is usually called the Littlewood-Richardson or intertwining number. O'Reilly (1982) recently derived such a formula:

$$[\lambda_1\mu_1]\otimes[\lambda_2\mu_2]=\sum_{k=0}^{\min(\mu_2,\lambda_1+\mu_1)}\ \sum_{j=0}^{\min(\mu_1,\lambda_2,\lambda_1+\mu_1-k)} \times \sum_{i=\max(0,j+k-\mu_1)}^{\min(\lambda_2-j+k,\lambda_1)}[\lambda_1+\lambda_2-j-2i+k,\ \mu_1+\mu_2+i-j-2k] \tag{2.2}$$

in terms of three nested summations. It will be shown herein that this formula contains a great deal of group theoretic information.

In order to provide a canonical resolution to the multiplicity problem, Biedenharn, Louck and collaborators, in a series of articles (see Louck 1970, Biedenharn *et al* 1972, Biedenharn and Louck 1972, Louck and Biedenharn 1973, Biedenharn *et al* 1985 and references therein), developed an elegant formalism in which they postulate the existence of an abstract set of SU(3) tensor operators classified by upper Gel'fand patterns (commonly referred to as operator patterns). They showed that this set of tensors can be denoted by

$$\mathscr{T}\begin{pmatrix}\Gamma\\ \{h\}\\ m\end{pmatrix} \tag{2.3}$$

where

$$/\Gamma\backslash = \begin{array}{c} \Gamma_{11} \\ \Gamma_{12} \qquad \Gamma_{22} \end{array} \tag{2.4}$$

is an operator pattern and the lower pattern

$$\backslash m/ = \begin{array}{c} m_{12} \qquad m_{22} \\ m_{11} \end{array} \tag{2.5}$$

labels a basis for the U(3) unirrep

$$\{h\}=\{h_1h_2h_3\}\equiv\{\lambda+\mu+\delta,\ \mu+\delta,\ \delta\} \tag{2.6}$$

(also irreducible under SU(3)) according to which this tensor operator transforms. The operator pattern characterises the structural and tensorial properties of $\mathscr{T}$, the most important being the following.

(i) The shifts

$$\Delta=\begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\end{pmatrix}=\begin{pmatrix}\Gamma_{11}\\ \Gamma_{12}+\Gamma_{22}-\Gamma_{11}\\ h_1+h_2+h_3-\Gamma_{12}-\Gamma_{22}\end{pmatrix} \tag{2.7}$$

indicate that, when applied to a state belonging to a U(3) unirrep $\{h'_1h'_2h'_3\}$ belonging to the Hilbert space, the tensor $\mathcal{T}$ will map this state to a new unirrep labelled by

$$\{h''_1h''_2h''_3\}=\{h'_1+\Delta_1,\, h'_2+\Delta_2,\, h'_3+\Delta_3\}. \tag{2.8}$$

(ii) The existence of $\mathscr{L}(\Delta)$ tensors having the same shifts Δ and identified by the condition

$$\Gamma_{12}+\Gamma_{22}=\text{constant} \tag{2.9}$$

in (2.7) indicates that a given unirrep $\{h''\}$ may appear more than once in the resolution of the corresponding Kronecker product $\{h\}\otimes\{h'\}$.

(iii) Some couplings predicted by the shift rule $\{h''\}=\{h'+\Delta\}$ to arise when the tensor is applied to a given state must vanish identically since we have $l\leqslant\mathscr{L}$.

(iv) The space $N(\Gamma^i)$ for which the application of an ordered set of tensors

$$\mathcal{T}\begin{pmatrix}\Gamma^i\\ \{h\}\\ m\end{pmatrix} \tag{2.10}$$

results in its annihilation is called the null space of the tensor. An inclusion property

$$N(\Gamma^1)\supset N(\Gamma^2)\supset\ldots\supset N(\Gamma^{\mathscr{L}}) \tag{2.11}$$

may be required to apply for such spaces.

(v) The tensors are required to be (orthogonal) unit tensors such that their matrix elements directly result in the definition of Wigner coefficients through the equality

$$\langle\{h^3\}m_3|\mathcal{T}\begin{pmatrix}\Gamma\\ \{h^3\}\\ m_2\end{pmatrix}|\{h^1\}m_1\rangle=\langle\{h^1\}m_1;\,\{h^2\}m_2|\{h^3\}m_3\rangle_\Gamma. \tag{2.12}$$

The identification of the multiplicity label with the operator patterns Γ then serves to emphasis that the Wigner coefficients thus defined inherit their structural properties from their parent tensors.

In the following, we give an explicit construction of a set of basic tensors acting on a Hilbert space of $\mathscr{U}(2)\otimes\mathrm{U}(3)$ Bargmann polynomials. A tensor in the set will be denoted

$$T^{(\gamma)\,[\lambda\mu]}_{\nu\;\;\;m} \tag{2.13}$$

where (γ) and $[\lambda\mu]$ indicate its $\mathscr{U}(2)$ and SU(3) ranks, and ν and m label a $\mathscr{U}(2)$ and SU(3) basis for (γ) and $[\lambda\mu]$. The tensors are constructed with the help of six Bargmann variables g_η and their derivatives $\partial/\partial g_\eta$ and, as such, each tensor has a uniquely defined U(3) rank $\{h\}$. Furthermore, it will be shown that the $\mathscr{U}(2)$ quantum numbers (γ) and ν provide a new interpretation of Biedenharn's operator pattern and endow the latter with an explicit group theoretical meaning.

It will be shown that the basic tensors (2.13) satisfy Biedenharn's conditions (i), (ii) and (iii) when their actions are restricted to a subspace (SU(3) model space) of the full Hilbert space. Furthermore, it will be shown that each term in the O'Reilly series (2.2) can be put in correspondence with a *unique* Bargmann tensor (2.13), a most gratifying result.

However, our tensors do not satisfy Biedenharn's conditions (iv) and (v). As a consequence, equation (2.12) must be replaced (after appropriate reduction with respect to the subgroup $\mathcal{U}(2)$, cf equation (6.2)) by the more general expression

$$\langle[\lambda_3\mu_3]m_3|T^{(\gamma)[\lambda_2\mu_2]}_{m_2}|[\lambda_1\mu_1]m_1\rangle = \sum_{\rho'=1}^{l}\langle[\lambda_1\mu_1]m_1;[\lambda_2\mu_2]m_2|[\lambda_3\mu_3]m_3\rangle_{\rho'}\langle[\lambda_3\mu_3]\|T^{(\gamma)[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho'} \quad (2.14)$$

in terms of both Wigner coefficients and reduced matrix elements (extended Wigner-Eckart theorem).

We will use a Gramm-Schmidt process to solve equation (2.14) for the Wigner coefficients. We therefore order the patterns $\gamma\rightarrow\gamma(\rho)$, $\rho=1,\ldots,l$ and impose

$$\langle[\lambda_3\mu_3]\|T^{(\gamma(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho'}=0 \qquad \text{if } \rho'>\rho. \quad (2.15)$$

The Wigner-Eckart theorem then gives

$$\langle[\lambda_3\mu_3]m_3|T^{(\gamma(\rho))[\lambda_2\mu_2]}_{m_2}|[\lambda_1\mu_1]m_1\rangle = \sum_{\rho'\leqslant\rho}\langle[\lambda_1\mu_1]m_1;[\lambda_2\mu_2]m_2|[\lambda_3\mu_3]m_3\rangle_{\rho'}\langle[\lambda_3\mu_3]\|T^{(\gamma(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho'}. \quad (2.16)$$

The left-hand side of this equation can be evaluated explicitly. The reduced matrix element $\langle[\lambda_3\mu_3]\|T^{(\gamma(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_\rho$ is shown to be a linear function of the Wigner coefficients $\langle[\lambda_1\mu_1]m_1;[\lambda_2\mu_2]m_2|[\lambda_3\mu_3]m_3\rangle_\rho$ for the same ρ which can all be deduced from a *single* chosen coefficient ($m_2=m_0$, m_1 and m_3 highest weights)

$$\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2\mu_2]m_0|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho$$

as shown in § 6. The algorithm then allows the computation of the quantity

$$\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2\mu_2]m_0|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho^2$$

from which all other coefficients can be obtained. The algorithm is simple enough to allow algebraic manipulations for low multiplicity cases as illustrated by an example in § 7.

For the Gramm-Schmidt process to be non-arbitrary, one must specify an unequivocal ordering of the tensors. Fortunately, the structure of our basic tensors is such that a unique ordering by null space inclusion properties can be given, as will be shown in § 8. Although our ordering does not duplicate property (iv) of Biedenharn's abstract set of tensors, it appears to reproduce an alternative resolution of the outer multiplicity problem first advocated by Hecht (1965).

First, recall that Hecht (1965) did derive recursion relations (see also Draayer and Akiyama 1973) which allow the computation of the values of generic Wigner coefficients

$$\langle[\lambda_1\mu_1]m_1;[\lambda_2\mu_2]m_2|[\lambda_3\mu_3]m_3\rangle_\rho \quad (2.17)$$

from a restricted set of Wigner coefficients of the form

$$\langle[\lambda_1\mu_1]m_1;[\lambda_2\mu_2]\mathrm{HW}|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho \quad (2.18a)$$

or

$$\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2\mu_2]m_2|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho. \quad (2.18b)$$

Now, it has been postulated (see, e.g., Braunschweig 1978) that the number of coefficients belonging to such restricted sets is equal to the intertwining number l. Equation (2.18a) would therefore define a square matrix $M_{\rho\rho'}$ where the ρ index refers to an ordered set of tensors and ρ' refers to an ordered set of allowed Gel'fand patterns $m_1(\rho')$. We show that this is the case for the very important class of self-conjugate SU(3) tensors (Lohe *et al* 1977) and, furthermore, that the structure of the Bargmann tensors *results* in M being strictly lower triangular. Now Hecht (1965) argued that one can resolve the SU(3) multiplicity problem simply by *requiring* M to be lower triangular. It is most satisfying to see that (at least our self-conjugate) Bargmann tensors duplicate such a simple resolution which 'does not suffer from the worst faults of an arbitrary labelling' (Hecht 1965). Furthermore, with the identification of the multiplicity index ρ with the upper patterns, Hecht's resolution becomes *ipso facto* equivalent to a canonical labelling.

In the more general case of coupling by non-self-conjugate tensors, we conjecture that M is still a square lower triangular matrix and give a partial proof by showing that M has null entries in its upper triangular part.

3. $\mathcal{U}(2)\otimes$U(3) Bargmann tensors

Following our previous developments (Le Blanc and Rowe 1986), we consider a Hilbert–Bargmann space of polynomials in six complex variables ($g_{\alpha i}$, $\alpha=1,2$, $i=1,2,3$). By complementarity (Biedenharn *et al* 1967), this space decomposes into unirreps $(h_1h_2)\otimes\{h_1h_2 0\}$ of the direct product group $\mathcal{U}(2)\otimes$U(3). Furthermore, this decomposition is multiplicity free. Thus a basis of Bargmann polynomials can be labelled by upper $\mathcal{U}(2)$ and lower U(3) Gel'fand patterns:

$$\langle g|[\lambda\mu]i;\, m\rangle \equiv \left\langle g \left| \begin{matrix} & & & \lambda+\mu-i & & & \\ \lambda+\mu & & \lambda+\mu & & \mu & & 0 \\ & & m_{12} & \mu & m_{22} & & \\ & & & m_{11} & & & \end{matrix} \right.\right\rangle \tag{3.1}$$

where the ranges of the parameters (i, m) are specified by the usual betweeness conditions. These polynomials are defined by their lowest weight ($i=0$, $m_{12}=m_{11}=\lambda+\mu$, $m_{22}=\mu$) components

$$\langle g|[\lambda\mu]\text{LW};\,\text{LW}\rangle = N[\lambda\mu]g_{11}^{\lambda}\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}^{\mu}. \tag{3.2a}$$

With the normalisation

$$N[\lambda\mu]=\left(\frac{\lambda+1}{(\lambda+\mu+1)!\,\mu!}\right)^{1/2} \tag{3.2b}$$

this basis becomes orthonormal with respect to the Bargmann measure. In terms of the two Bargmann vectors, the u(3) Lie algebra is then given (with summation over repeated indices) by

$$C_{ij}=g_{\alpha i}\frac{\partial}{\partial g_{\alpha j}} \tag{3.3}$$

while the Lie algebra of the complementary group $\mathcal{U}(2)$ is given by

$$\mathscr{C}_{\alpha\beta}=g_{\alpha i}\frac{\partial}{\partial g_{\beta i}}. \tag{3.4}$$

Note that, within the Bargmann space thus defined, each SU(3) representation appears with a multiplicity equal to the dimension $(\lambda+1)$ of the corresponding $\mathscr{S}\mathscr{U}(2)$ representation $(\lambda/2)$ (in the usual angular momentum notation). Thus the full Bargmann space is not a model space for SU(3) (Bernšteĭn *et al* 1975). But if a restriction is made to states that are lowest weight with respect to the complementary group $\mathscr{U}(2)$, then such a model space is defined.

To give a realisation of Biedenharn's set of tensor operators on this Bargmann space, it is first necessary to unambiguously select the appropriate upper patterns (Le Blanc and Rowe 1986). Note that the Bargmann polynomials have at most two rows in their corresponding Young tableaux and it is therefore neceessary to impose the condition $\Delta_3=0$ on the upper patterns (2.4). This can be done by first considering the double Gel'fand pattern (2.3) where h_3 is set to zero:

$$\left\langle \begin{matrix} & & & \lambda+\mu-j-i & & & \\ & \lambda+\mu-j & & & & \mu-k & \\ \lambda+\mu & & & \mu & & & 0 \\ & m_{12} & & & & m_{22} & \\ & & & m_{11} & & & \end{matrix} \right\rangle \tag{3.5a}$$

and where the spans for (ijk) are given by (betweeness conditions)

$$0\leqslant j\leqslant\lambda \qquad 0\leqslant k\leqslant\mu \qquad 0\leqslant i\leqslant\lambda-j+k. \tag{3.5b}$$

This enumeration is strictly equivalent to Biedenharn's enumeration. According to (2.7), the Δ_3 shift associated with the pattern (3.5) is given by

$$\delta=(j+k). \tag{3.6}$$

By subtracting δ across the whole pattern (3.5a),

$$\left\langle \begin{matrix} & & & \lambda+\mu-j-i-\delta & & & \\ & \lambda+\mu-j-\delta & & & & \mu-k-\delta & \\ \lambda+\mu-\delta & & & \mu-\delta & & & -\delta \\ & m_{12}-\delta & & & & m_{22}-\delta & \\ & & & m_{11}-\delta & & & \end{matrix} \right\rangle \tag{3.7a}$$

with, as before,

$$0\leqslant j\leqslant\lambda \qquad 0\leqslant k\leqslant\mu \qquad 0\leqslant i\leqslant\lambda-j+k \tag{3.7b}$$

we obtain the appropriate pattern for which the associated Δ_3 shift does indeed now vanish, as can be easily verified. The pattern (3.7) will unambiguously set the $\mathscr{U}(2)\otimes$ U(3) tensorial properties of the Bargmann tensor operators built below.

The use of the indices (ijk) in the enumeration (3.7b) is intentionally the same as those used in the O'Reilly series (2.2) for reasons that will become clearer in the following. We stress the fact that in our construction, the pattern (3.7) labels tensors that are irreducible under the group product $\mathscr{U}(2)\otimes$U(3). The upper pattern

$$\left/ \begin{matrix} & \lambda+\mu-2j-k-i & \\ \lambda+\mu-2j-k & & \mu-j-2k \end{matrix} \right\backslash \tag{3.8}$$

refers to the $\mathscr{U}(2)$ rank

$$(\gamma_1,\gamma_2)\equiv(\lambda+\mu-2j-k,\mu-j-2k) \tag{3.9a}$$

and the $\mathscr{U}(1)$ weight

$$\nu\equiv\lambda+\mu-2j-k-i \tag{3.9b}$$

of the tensor while the lower pattern refers to its U(3) rank

$$\{h_1h_2h_3\}=\{\lambda+\mu-(j+k),\mu-(j+k),-(j+k)\} \tag{3.10}$$

and the corresponding Gel'fand basis. For this reason we denote the tensors of equation (2.3) by the more suggestive notation

$$T^{(\gamma_1\gamma_2)}_{\nu}{}^{[\lambda\mu]}_{m}. \tag{3.11}$$

The Bargmann tensor (3.11) is defined by its $\mathscr{U}(2)\otimes \mathrm{U}(3)$ lowest weight component

$$T^{(\gamma_1\gamma_2)}_{\mathrm{lw}}{}^{[\lambda\mu]}_{\mathrm{lw}} = (T^{(11)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}})^{\mu-k}(T^{(10)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}})^{\lambda-j}(T^{(0-1)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}})^{k}(T^{(-1-1)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}})^{j} \tag{3.12}$$

in terms of the four elementary $\mathscr{U}(2)\otimes \mathrm{U}(3)$ tensor operators

$$T^{(10)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}}(g) = g_{11} \qquad T^{(-1-1)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}}(g) = (\partial_1 \wedge \partial_2)_1 \tag{3.13a}$$

and

$$T^{(0-1)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}}(g) = \partial_{23} = \partial/\partial g_{23} \qquad T^{(11)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}}(g) = (g_1 \wedge g_2)_3. \tag{3.13b}$$

When $j=k=0$, the tensor (3.12) simply reduces to an unnormalised $\mathscr{U}(2)\otimes \mathrm{U}(3)$ Bargmann polynomial (3.1) whose double Gel'fand pattern is of the form (3.7) with $\delta=0$.

The tensors (3.12) have been given a normal ordering in terms of creation and annihilation operators. It can be verified that only the components $T^{(11)[01]}$ and $T^{(-1-1)[10]}$ do not commute in (3.12) and that their commutator is given by

$$[T^{(11)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}}, T^{(-1-1)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}}] = C_{\mathrm{lw}} = C_{13} \tag{3.14}$$

where C_{13} is now commuting with all the basic operators (3.13). In our previous paper (Le Blanc and Rowe 1986), it was argued that one should replace the product

$$(T^{(11)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}} \times T^{(-1-1)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}})^{e} \qquad e = \min(\mu-k, j) \tag{3.15}$$

in equation (3.12) with C_{13}^{e} in order to maintain consistency with the abstract sets of tensors used by Louck and Biedenharn (1970) and Draayer and Akiyama (1973). However, we now find that we can make more progress and obtain a correspondence with the O'Reilly formula if we retain the normal ordered definition of equation (3.12). However, since the elementary Bargmann tensors $T^{(10)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}}$ and $T^{(0-1)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}}$ commute, the lowest weight components of equation (3.12) can also be expressed

$$T^{(\gamma_1\gamma_2)}_{\mathrm{lw}}{}^{[\lambda\mu]}_{\mathrm{lw}} = (T^{(11)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}})^{\mu-k}(T^{(0-1)}_{\mathrm{lw}}{}^{[01]}_{\mathrm{lw}})^{k}(T^{(10)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}})^{\lambda-j}(T^{(-1-1)}_{\mathrm{lw}}{}^{[10]}_{\mathrm{lw}})^{j}. \tag{3.16}$$

Also recall that a tensor is uniquely defined by its lowest weight component. We thus consider the $\mathscr{U}(2)\otimes \mathrm{U}(3)$ tensors

$$T^{(\gamma_1\gamma_2)[\lambda\mu]} = [T^{(\mu-k,\mu-2k)[0\mu]} \times T^{(\lambda-2j,-j)[\lambda 0]}]^{(\gamma_1\gamma_2)[\lambda\mu]} \tag{3.17}$$

obtained from the stretched coupling of

$$T^{(\mu-k,\mu-2k)[0\mu]} = (T^{(11)[01]})^{\mu-k}(T^{(0-1)[01]})^{k} \tag{3.18a}$$

and

$$T^{(\lambda-2j,-j)[\lambda 0]} = (T^{(10)[10]})^{\lambda-j}(T^{(-1-1)[10]})^{j}. \tag{3.18b}$$

4. Null spaces

When a Bargmann tensor (3.17) acts on a Bargmann polynomial (3.1), not all polynomials enumerated in the O'Reilly series will be reached by the action of the tensor as some selection rules will intervene in addition to the usual rules arising from the additivity of the $\mathscr{U}(2)$ and U(3) weights. Due to its $\mathscr{SU}(2)$ tensorial properties, the tensor $T^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ will map states belonging to an SU(3) unirrep $[\lambda_1\mu_1]$ to (at most) a set of unirreps $[\lambda_3\mu_3]$ such that

$$|J_1-J_2|\leq J_3\leq J_1+J_2 \tag{4.1a}$$

with

$$J_1=\lambda_1/2 \qquad J_3=\lambda_3/2 \qquad J_2=(\gamma_1-\gamma_2)/2=(\lambda_2-j+k)/2. \tag{4.1b}$$

Also, from the additivity of the eigenvalues of the $\mathscr{U}(2)$ trace operator (also trace operator for U(3))

$$\mathscr{C}_{\alpha\alpha}=C_{ii} \tag{4.2}$$

we have

$$\lambda_1+2\mu_1+\lambda_2+2\mu_2-3j-3k=\lambda_3+2\mu_3. \tag{4.3}$$

We obtain from (4.1) and (4.3) that the Bargmann tensor $T^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ will map an SU(3) unirrep $[\lambda_1\mu_1]$ to (at most) the unirreps

$$T^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}:[\lambda_1\mu_1]\to\sum_{i=0}^{\min(\lambda_1,\lambda_2-j+k)}[\lambda_1+\lambda_2-j+k-2i,\ \mu_1+\mu_2-j-2k+i]. \tag{4.4}$$

Let us now denote by $\mathscr{T}_\nu^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ the restriction of the U(3) tensor $T_\nu^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ to the model subspace of $\mathscr{U}(2)$ lowest weight Bargmann polynomials, i.e. the polynomials with $i=0$ in equation (3.1). On restriction to this subspace, we have from the additivity of the $\mathscr{U}(1)$ weights that

$$\lambda_1+\mu_1+\lambda_2+\mu_2-2j-i-k=\lambda_3+\mu_3. \tag{4.5}$$

From (4.3) and (4.5), it is verified that equation (4.4) then reduces to

$$\mathscr{T}_\nu^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}:[\lambda_1\mu_1]\to[\lambda_1+\lambda_2-j+k-2i,\ \mu_1+\mu_2-j-2k+i] \tag{4.6}$$

from which it follows that the tensor $\mathscr{T}_\nu^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ has shift properties for λ and μ given by

$$\begin{aligned}\lambda_3&=\lambda_1+\lambda_2-j+k-2i\\ \mu_3&=\mu_1+\mu_2-j-2k+i.\end{aligned} \tag{4.7}$$

These are precisely the shifts predicted by Biedenharn if the upper ($\mathscr{U}(2)$) pattern is interpreted as an operator pattern (see equations (2.7) and (3.8)). We thus conclude that the Bargmann tensors (3.12) have precisely the shifts assigned to them by their operator patterns when their actions are restricted to the model space.

Equation (4.6) strongly suggests that each term in the O'Reilly series (2.2) can be related to a unique Bargmann tensor (3.17). But the ranges of the parameters (ijk) appear to be different. This is because, when acting on a particular $[\lambda_1\mu_1]$ polynomial, some of the Bargmann tensors (3.17) will return identically zero. The states that are annihilated by a particular tensor constitute what is called its *null space*. We now show that if i, j or k lie outside the ranges given by the O'Reilly formula, the initial $[\lambda_1\mu_1]$ polynomials are in the null space of the corresponding tensor.

Since the null space properties of a Bargmann tensor (3.17) arise as much from its detailed structure as from its general tensorial properties, we will examine the actions of its constituent parts given by the general formula (4.4). The restrictions on the indices i, j and k will be dealt with afterwards.

(*a*) Application of the rightmost term in (3.16) maps $[\lambda_1\mu_1]$ to

$$(T^{(-1-1)[10]})^j : [\lambda_1\mu_1] \to [\lambda_1, \mu_1 - j]. \tag{4.8}$$

(*b*) The application of the two following terms on $[\lambda_1, \mu_1 - j]$ leads to (usual $\mathscr{U}(2)$ product rule)

$$(T^{(0-1)[01]})^k (T^{(10)[10]})^{\lambda_2-j} : [\lambda_1\mu_1 - j] \to \sum_{i=0}^{\min(\lambda_1,\lambda_2-j+k)} [\lambda_1+\lambda_2-j+k-2i, \mu_1-j-k+i]. \tag{4.9}$$

(*c*) Finally, application of the leftmost term in (3.16) leads to

$$(T^{(11)[01]})^{\mu_2-k} : \sum_i [\lambda_1+\lambda_2-j+k-2i, \mu_1-j-k+i] \to \sum_i [\lambda_1+\lambda_2-j+k-2i, \mu_1+\mu_2-j-2k+i]. \tag{4.10}$$

We now examine the allowed spans for i, j and k.

(*a*) For fixed j and k, i will have its normal span $0 \leq i \leq \lambda_2 - j + k$ given by equation (3.7*b*) with the additional restrictions given by equation (4.9)

$$j + k - \mu_1 \leq i \leq \min(\lambda_2 - j + k, \lambda_1)$$

where the lower limit prevents μ from assuming a negative value. Thus i is restricted to the range

$$\max(0, j+k-\mu_1) \leq i \leq \min(\lambda_2 - j + k, \lambda_1). \tag{4.11}$$

(*b*) For fixed k, j will have the normal span $0 \leq j \leq \lambda_2$ given by equation (3.7*b*) with the additional restriction $j \leq \mu_1$ from the mapping (4.8) and with the further restriction $j \leq \lambda_1 + \mu_1 - k$ coming from the fact that the term $(T^{(0-1)[01]})^k$ would annihilate the intermediate representation $[\lambda_1, \mu_1 - j]$ for $j > \lambda_1 + \mu_1 - k$. Thus j is restricted to the range

$$0 \leq j \leq \min(\lambda_2, \mu_1, \lambda_1 + \mu_1 - k). \tag{4.12}$$

(*c*) Finally, k will have the normal span $0 \leq k \leq \mu_2$ given by equation (3.7*b*) with the only additional restriction $k \leq \lambda_1 + \mu_1$, otherwise the term $(T^{(0-1)[01]})^k$ would annihilate the initial representation $[\lambda_1\mu_1]$. Thus k is restricted to the range

$$0 \leq k \leq \min(\mu_2, \lambda_1 + \mu_1). \tag{4.13}$$

Once more we recall that, when restricted to the model space, the U(3) tensor $\mathscr{T}_\nu^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ maps $[\lambda_1\mu_1]$ to a unique $[\lambda_3\mu_3]$ according to the shift rules (4.7) (i.e. there is no more summation on i in (4.9) and (4.10)); i assumes the specific value given by (4.7) (for fixed k and j).

We thus have obtained the very interesting result that, when its action is restricted to the model space, each Bargmann tensor $\mathscr{T}_\nu^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}$ corresponds to a unique term in the O'Reilly series. More important, if either i, j, or k lies out of the bounds allowed by the summation limits of the O'Reilly formula, the corresponding tensor has vanishing matrix elements. In other words, there are exactly l and no more $[\lambda_2\mu_2]$ Bargmann tensors (3.17) with non-vanishing matrix elements between given $[\lambda_1\mu_1]$ and $[\lambda_3\mu_3]$

representations. Thus we conclude that properties (i), (ii) and (iii) hold for the Bargmann tensors (3.17).

A multiplicity set is defined by the set of all the combinations of the indices (ijk) leading to the same final representation $[\lambda_3\mu_3]$ in (6.3). We easily see that these are related by

$$
\begin{aligned}
k(\rho) &= k_{\min} + (\rho - 1) \\
j(\rho) &= j_{\max} - (\rho - 1) \qquad 1 \leqslant \rho \leqslant l \\
i(\rho) &= i_{\min} + (\rho - 1)
\end{aligned}
\tag{4.14}
$$

where $k_{\min}$, $j_{\max}$ and $i_{\min}$ are the maximal or minimal values allowed by the limits in the O'Reilly series. Note that the index ρ orders the tensors belonging to a multiplicity set and that the Gramm–Schmidt process for the calculation of the Wigner coefficients which will be expounded in § 6 will respect this ordering. It will be shown in § 8 that the ordering (4.14) is *not* arbitrary but rather has been chosen such that a subset of a restricted set of Wigner coefficients would conveniently vanish as discussed in § 2.

5. Matrix elements for the $[\lambda 0]$ and $[\mu 0]$ tensors

It is clear from equation (2.14) that, from a knowledge of the reduced matrix elements of generic Bargmann tensors, one can easily deduce the corresponding Wigner coefficients. The reduced matrix elements are calculated most easily when either λ_2 or $\mu_2 = 0$ because the coupling is then multiplicity free. Expressions for these reduced matrix elements were previously given by us and will be recalled here because they play a fundamental role in determining the reduced matrix elements for a generic Bargmann tensor which has been factored according to equation (3.17).

Matrix elements of the tensor (3.18a) have been determined in Le Blanc and Rowe (1986). We found that, with their help, analytical expressions could be given for the SU(3) isoscalar factors

$$
\begin{aligned}
\langle [\lambda_1\mu_1]\text{HW}; [\lambda_2 0]\varepsilon_2\Lambda_2 \| [\lambda_3\mu_3]\text{HW}\rangle^2 &= \lambda_1!(\lambda_3+1)!(\lambda_1+\mu_1+1)!(\lambda_3+\mu_3+2)! \\
&\quad \times \{[\tfrac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1+2\mu_3+3)]![\tfrac{1}{3}(\lambda_1+\lambda_2+2\lambda_3-\mu_1+\mu_3+3)]! \\
&\quad \times [\tfrac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]![\tfrac{1}{3}(\lambda_1+\lambda_2+2\lambda_3+2\mu_1+\mu_3+6)]!\}^{-1}
\end{aligned}
\tag{5.1a}
$$

where (Hecht 1965)

$$
\begin{aligned}
\varepsilon_2 &= 2\lambda_3 + \mu_3 - 2\lambda_1 - \mu_1 \\
\Lambda_2 &= \tfrac{1}{3}(2\lambda_2 - \varepsilon_2).
\end{aligned}
\tag{5.1b}
$$

(We will agree to take a $(+1)$ phase factor in the following when taking the square root of the right-hand side of (5.1a).) Other Wigner coefficients pertaining to this coupling can be obtained with the use of recursion formulae (Hecht 1965, Draayer and Akiyama 1973, but see also Fujiwara and Horiuchi 1983).

The $\mathcal{U}(2)\otimes U(3)$ reduced matrix elements of tensor (3.18*a*) were also calculated and found to be given by

$$\frac{\langle[\lambda_3\mu_3]\| T^{(\lambda_2-2j,-j)[\lambda_2 0]}\|[\lambda_1\mu_1]\rangle}{\langle[\lambda_1\mu_1]\mathrm{HW};\,[\lambda_2 0]\varepsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle}$$

$$=\left(\frac{[\frac{1}{3}(\lambda_1+\lambda_2+2\lambda_3+2\mu_1+\mu_3+6)]!}{(\lambda_3+\mu_3+2)(\lambda_3+1)}\right)$$

$$\times\{[\tfrac{1}{3}(2\lambda_3+\lambda_1+\lambda_2-\mu_1+\mu_3+3)]!$$

$$\times[\tfrac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1-2\mu_3)]![\tfrac{1}{3}(2\lambda_1-\lambda_2+\lambda_3+\mu_1-\mu_3)]!\}^{1/2}$$

$$\times\left(\frac{(\lambda_1+1)[\frac{1}{3}(2\lambda_2-\lambda_1+\lambda_3-2\mu_1+2\mu_3)]![\frac{1}{3}(\lambda_1+\lambda_2-\lambda_3+2\mu_1+\mu_3+3)]!}{(\mu_3+1)\lambda_1!\lambda_2!\lambda_3!(\lambda_1+\mu_1+1)!(\lambda_3+\mu_3+1)![\frac{1}{3}(\lambda_3-\lambda_1-\lambda_2+\mu_1+2\mu_3)]!}\right)^{1/2} \tag{5.2}$$

i.e. in the Bargmann space, the doubly reduced matrix element is proportional to the SU(3) isoscalar coefficient (5.1*a*) as mentioned in § 2.

Wigner coefficients pertaining to the coupling by a $[0\mu_2]$ tensor are easily obtained by Hermiticity considerations (Hecht 1965, Draayer and Akiyama 1973). The doubly reduced matrix elements $\langle[\lambda_3\mu_3]\| T^{(\gamma_1\gamma_2)[0\mu_2]}\|[\lambda_1\mu_1]\rangle$ are then given by

$$\langle[\lambda_3\mu_3]\| T^{(\gamma_1\gamma_2)[0\mu_2]}\|[\lambda_1\mu_1]\rangle$$

$$=(-1)^{(\lambda_1+\gamma_1-\gamma_2-\lambda_3)/2}\left(\frac{\lambda_1+1}{\lambda_3+1}\right)^{1/2}(-1)^{\varphi}\left(\frac{\dim[\lambda_1\mu_1]}{\dim[\lambda_3\mu_3]}\right)^{1/2}$$

$$\times\langle[\lambda_1\mu_1]\| T^{(\gamma_2\gamma_1)[\mu_2 0]}\|[\lambda_3\mu_3]\rangle \tag{5.3a}$$

with

$$\varphi=\lambda_1+\lambda_2-\lambda_3+\mu_1+\mu_2-\mu_3. \tag{5.3b}$$

Since matrix elements of the tensors (3.18) are given in closed form, a relatively simple expression is readily obtained for the matrix elements of the generic composite tensor (3.17) (cf equation (6.2)).

6. Computation of the Wigner coefficients

Following the strategy outlined in § 2, we first evaluate the matrix elements

$$\langle[\lambda_3\mu_3]i_3;\,\varepsilon_3\Lambda_3M_{\Lambda_3}| T_{\nu}^{(\gamma_1(\rho)\gamma_2(\rho))}{}^{[\lambda_2\mu_2]}_{\varepsilon_2\Lambda_2M_{\Lambda_2}}\|[\lambda_1\mu_1]i_1;\,\varepsilon_1\Lambda_1M_{\Lambda_1}\rangle \tag{6.1}$$

of the basic tensors (3.11) where we use Hecht's notation for the SU(3) basis (Hecht 1965) (see also equation (8.1)). These are easily derived for $\varepsilon_1\Lambda_1M_{\Lambda_1}$ and $\varepsilon_3\Lambda_3M_{\Lambda_3}$ highest weights using the factorisation (3.17) and the results are summarised in terms of the $\mathscr{S}\mathscr{U}(2)\otimes\mathrm{SU}(2)$ doubly reduced matrix elements

$$
\begin{aligned}
&\langle[\lambda_3\mu_3]\mathrm{HW}|\, T^{(\gamma_1(\rho)\gamma_2(\rho))\,[\lambda_2\mu_2]}_{\varepsilon_2\Lambda_2}|[\lambda_1\mu_1]\mathrm{HW}\rangle \\
&\quad = \sum_{\substack{[\lambda\mu]\varepsilon\Lambda\\ \varepsilon_{\lambda_2}\Lambda_{\lambda_2}\varepsilon_{\mu_2}\Lambda_{\mu_2}}} U\left(\frac{\lambda_1}{2}\,\frac{\lambda_2-j(\rho)}{2}\,\frac{\lambda_3}{2}\,\frac{k(\rho)}{2};\,\frac{\lambda}{2}J_2(\rho)\right) U\left(\frac{\mu_1}{2}\Lambda_{\lambda_2}\,\frac{\mu_3}{2}\Lambda_{\mu_2};\,\Lambda\Lambda_2\right) \\
&\qquad \times\langle[\lambda_2 0]\varepsilon_{\lambda_2}\Lambda_{\lambda_2};\,[0\mu_2]\varepsilon_{\mu_2}\Lambda_{\mu_2}\|[\lambda_2\mu_2]\varepsilon_2\Lambda_2\rangle \\
&\qquad \times\langle[\lambda_1\mu_1]\mathrm{HW};\,[\lambda_2 0]\varepsilon_{\lambda_2}\Lambda_{\lambda_2}\|[\lambda\mu]\varepsilon\Lambda\rangle\langle[\lambda\mu]\|\, T^{(\lambda_2-2j(\rho),j(\rho))[\lambda_2 0]}\|[\lambda_1\mu_1]\rangle \\
&\qquad \times\langle[\lambda\mu]\varepsilon\Lambda;\,[0\mu_2]\varepsilon_{\mu_2}\Lambda_{\mu_2}\|[\lambda_3\mu_3]\mathrm{HW}\rangle\langle[\lambda_3\mu_3]\|\, T^{(\mu_2-k(\rho),\mu_2-2k(\rho))[0\mu_2]}\|[\lambda\mu]\rangle
\end{aligned}
\tag{6.2}
$$

where $k(\rho)$ and $j(\rho)$ are given by (4.14) and $J_2(\rho)$ is given by (4.1b). The sum on the right-hand side is easily performed as all its components are known analytically (cf § 5, Le Blanc and Rowe 1986, Fujiwara and Horiuchi 1983), thus strongly favouring the choice of factorisation (3.17) for the Bargmann tensors (3.11).

Following equation (2.16), the equation for the Wigner coefficients becomes

$$
\begin{aligned}
&\sum_{\rho'\leqslant\rho}\langle[\lambda_1\mu_1]\mathrm{HW};\,[\lambda_2\mu_2]\varepsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle_{\rho'}\langle[\lambda_3\mu_3]\|\, T^{(\gamma_1(\rho)\gamma_2(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho'} \\
&\quad =\langle[\lambda_3\mu_3]\mathrm{HW}|\, T^{(\gamma_1(\rho)\gamma_2(\rho))\,[\lambda_2\mu_2]}_{\varepsilon_2\Lambda_2}|[\lambda_1\mu_1]\mathrm{HW}\rangle \\
&\quad \equiv\mathscr{P}(\Lambda_2,\rho).
\end{aligned}
\tag{6.3}
$$

From the triangular form ($\rho'\leqslant\rho$) of this equation, we therefore obtain

$$
\begin{aligned}
&\langle[\lambda_1\mu_1]\mathrm{HW};\,[\lambda_2\mu_2]\varepsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle_{\rho}\langle[\lambda_3\mu_3]\|\, T^{(\gamma_1(\rho)\gamma_2(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho} \\
&\quad =\mathscr{P}(\Lambda_2,\rho)-\mathscr{O}(\Lambda_2,\rho)\equiv\mathscr{Q}(\Lambda_2,\rho)
\end{aligned}
\tag{6.4}
$$

where

$$
\begin{aligned}
\mathscr{O}(\Lambda_2,\rho)\equiv&\sum_{\rho'\leqslant\rho-1}\langle[\lambda_1\mu_1]\mathrm{HW};\,[\lambda_2\mu_2]\varepsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle_{\rho'} \\
&\quad \times\langle[\lambda_3\mu_3]\|\, T^{(\gamma_1(\rho)\gamma_2(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho'} \\
=&\,0 \qquad \text{for } \rho=1
\end{aligned}
\tag{6.5}
$$

is the contribution to the ρth tensor of the $\rho'\leqslant\rho-1$ couplings.

To solve this equation, we start by expanding the $\mathscr{S}\mathscr{U}(2)\otimes\mathrm{SU}(3)$ doubly reduced matrix element

$$
\begin{aligned}
&\langle[\lambda_3\mu_3]\|\, T^{(\gamma_1(\rho)\gamma_2(\rho))[\lambda_2\mu_2]}\|[\lambda_1\mu_1]\rangle_{\rho'} \\
&\quad =\sum_{[\lambda\mu]} U\left(\frac{\lambda_1}{2}\,\frac{\lambda_2-j(\rho)}{2}\,\frac{\lambda_3}{2}\,\frac{k(\rho)}{2};\,\frac{\lambda}{2}J_2(\rho)\right) \\
&\qquad \times U([\lambda_1\mu_1][\lambda_2 0][\lambda_3\mu_3][0\mu_2];\,[\lambda\mu]_{--}[\lambda_2\mu_2]_{-}\rho') \\
&\qquad \times\langle[\lambda_3\mu_3]\|\, T^{(\mu_2-k(\rho),\mu_2-2k(\rho))[0\mu_2]}\|[\lambda\mu]\rangle\langle[\lambda\mu]\|\, T^{(\lambda_2-2j(\rho),j(\rho))[\lambda_2 0]}\|[\lambda_1\mu_1]\rangle
\end{aligned}
\tag{6.6}
$$

where $U([\lambda_1\mu_1][\lambda_2 0][\lambda_3\mu_3][0\mu_2];[\lambda\mu]_{--}[\lambda_2\mu_2]_{-}\rho')$ is an SU(3) Racah coefficient which satisfies

$$\langle[\lambda\mu]\mathrm{HW};[0\mu_2]\varepsilon'\Lambda'\|[\lambda_3\mu_3]\mathrm{HW}\rangle U([\lambda_1\mu_1][\lambda_2 0][\lambda_3\mu_3][0\mu_2];[\lambda\mu]_{--}[\lambda_2\mu_2]_{-}\rho')$$
$$=\sum_{\varepsilon\Lambda\Lambda_1\Lambda_2}(-1)^{\Lambda_2-\Lambda-\Lambda'}\langle[0\mu_2]\varepsilon'\Lambda';[\lambda_2 0]\varepsilon\Lambda\|[\lambda_2\mu_2]\varepsilon_2\Lambda_2\rangle U\left(\Lambda_1\Lambda\frac{\mu_3}{2}\Lambda';\frac{\mu}{2}\Lambda_2\right)$$
$$\times\langle[\lambda_1\mu_1]\varepsilon_1\Lambda_1;[\lambda_2 0]\varepsilon\Lambda\|[\lambda\mu]\varepsilon\Lambda\rangle\langle[\lambda_1\mu_1]\varepsilon_1\Lambda_1;[\lambda_2\mu_2]\varepsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle_{\rho'} \tag{6.7}$$

(Hecht 1965). Furthermore, the general Wigner coefficients

$$\langle[\lambda_1\mu_1]\varepsilon_1\Lambda_1;[\lambda_2\mu_2]\varepsilon_2\Lambda_2\|[\lambda_2\mu_3]\mathrm{HW}\rangle_{\rho'}$$

appearing in the latter can all be related to the coefficient (2.18*b*) with, say, Λ_2 minimal, by taking the ratios

$$\frac{\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2\mu_2]\varepsilon_2\Lambda_2\|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho}{\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2\mu_2]\varepsilon_2\Lambda_{2_{\min}}\|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho}=\frac{\mathscr{Q}(\Lambda_2,\rho)}{\mathscr{Q}(\Lambda_{2_{\min}},\rho)} \tag{6.8}$$

and introducing these ratios in the recursion formulae derived by Hecht (1965). Thus the doubly reduced matrix element (6.6) becomes proportional to

$$x=\langle[\lambda_1\mu_1]\mathrm{HW};[\lambda_2\mu_2]\varepsilon_2\Lambda_{2_{\min}}\|[\lambda_3\mu_3]\mathrm{HW}\rangle_\rho \tag{6.9}$$

and equation (6.4) reduces to an explicit expression for x^2. General Wigner coefficients can then be derived from (6.8) and the recursion formulae given by Hecht (1965) and Draayer and Akiyama (1973).

To obtain $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ Wigner coefficients, one would introduce on the left-hand side of (6.2) the now known values of the $\mathscr{S}\mathscr{U}(2)\otimes\mathrm{SU}(3)$ reduced matrix elements. Then the right-hand side of (6.2) would be recast in a form appropriate to the $\mathrm{SU}(3)\downarrow\mathrm{SO}(3)$ reduction by providing values for the multiplicity-free coefficients

$$\langle[\lambda_1\mu_1](\delta_1)L_1;[\lambda_2 0]L_2\|[\lambda_3\mu_3](\delta_3)L_3\rangle \tag{6.10a}$$

and (using Hermiticity considerations (Vergados 1968))

$$\langle[\lambda_1\mu_1](\delta_1)L_1;[0\mu_2]L_2\|[\lambda_3\mu_3](\delta_3)L_3\rangle \tag{6.10b}$$

which can be obtained in a recursive fashion from matrix elements of the elementary tensors (3.18) in a canonical $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ basis:

$$\langle g|[\lambda\mu](\delta)LM\rangle \tag{6.11}$$

(Le Blanc and Rowe 1985a, b, 1986). We intend to address this problem more thoroughly in a future publication.

7. An example

We will illustrate the algorithm of § 6 for the case of the coupling of an SU(3) unirrep $[\lambda\mu]$ to itself by a [11] tensor. For $\lambda,\mu\geqslant 1$, this coupling has multiplicity $l=2$. According to (3.7) and (4.14), the Bargmann tensors needed are $T^{(00)[11]}$ and $T^{(1-1)[11]}$, *in this order.*

We have from (6.2) and (6.8) that

$$\frac{\langle[\lambda]\mathrm{HW};[11]01\|[\lambda\mu]\mathrm{HW}\rangle_1}{\langle[\lambda\mu]\mathrm{HW};[11]00\|[\lambda\mu]\mathrm{HW}\rangle_1}=\frac{U(\frac{1}{2}\mu,\frac{1}{2},\frac{1}{2}\mu,\frac{1}{2};\frac{1}{2}(\mu-1),1)}{U(\frac{1}{2}\mu,\frac{1}{2},\frac{1}{2}\mu,\frac{1}{2};\frac{1}{2}(\mu-1),0)}\frac{\langle[10]-1\frac{1}{2};[01]1\frac{1}{2}\|[11]01\rangle}{\langle[10]-1\frac{1}{2};[01]1\frac{1}{2}\|[11]00\rangle}$$

$$=-\left(\frac{3(\mu+2)}{\mu}\right)^{1/2}. \tag{7.1}$$

Defining

$$\langle[\lambda\mu]\mathrm{HW};[11]00\|[\lambda\mu]\mathrm{HW}\rangle_1=x$$

we find with the help of (7.1) and the recursion formulae given by Hecht (1965) that

$$\begin{aligned}
&\langle[\lambda\mu]\mathrm{HW};[11]01\|[\lambda\mu]\mathrm{HW}\rangle_1=-\left(\frac{3(\mu+2)}{\mu}\right)^{1/2}x\\
&\left\langle[\lambda\mu]2\lambda+\mu-3\frac{\mu-1}{2};[11]3\tfrac{1}{2}\|\lambda\mu]\mathrm{HW}\right\rangle_1=-\left(\frac{6(\mu+1)}{\mu(\lambda+\mu+1)}\right)^{1/2}x\\
&\left\langle[\lambda\mu]2\lambda+\mu-3\frac{\mu+1}{2};[11]3\tfrac{1}{2}\|[\lambda\mu]\mathrm{HW}\right\rangle_1=0.
\end{aligned} \tag{7.2}$$

Note the vanishing of the last coefficient as can be *predicted* from § 8. When (7.2) is introduced into (6.7), we find

$$U([\lambda\mu][10][\lambda\mu][01];[\lambda\mu-1]_{--}[11]_{-1})$$
$$=-\left(\frac{2}{3\mu(\mu+1)}\right)^{1/2}\frac{2\mu\lambda+2\mu^2+8\mu+3\lambda+6}{[(\lambda+\mu+1)(\lambda+\mu+2)]^{1/2}}x \tag{7.3}$$

which, with the help of (6.4) and (6.6), leads to

$$x^2=\frac{\mu(\lambda+\mu+1)}{2(2\mu\lambda+2\mu^2+8\mu+3\lambda+6)}. \tag{7.4}$$

It is natural to take a (+1) phase when taking the square root in (7.4). However, any other phase convention could equally well be adopted. One can easily verify that the normalisation (7.4) is correct.

Using the first coupling thus defined, we then find from (6.7)

$$\begin{aligned}
&U([\lambda\mu][10][\lambda\mu][01];[\lambda+1\mu]_{--}[11]_{-1})=\left(\frac{2(\lambda+2)(\lambda+\mu+3)}{3(\lambda+1)(\lambda+\mu+2)}\right)^{1/2}x\\
&U([\lambda\mu][10][\lambda\mu][01];[\lambda-1\mu+1]_{--}[11]_{-1})=-\left(\frac{2\lambda(\mu+2)}{3(\lambda+1)(\mu+1)}\right)^{1/2}x
\end{aligned} \tag{7.5}$$

from which we deduce

$$\langle[\lambda\mu]\|T^{(1-1)[11]}\|[\lambda\mu]\rangle_1=-[\tfrac{1}{3}\lambda(\lambda+2)]^{1/2}x \tag{7.6}$$

using (6.6).

We obtain from (6.2) and (6.3)

$$\mathscr{P}(\Lambda_2=1,\rho=2)=-\frac{1}{2}\left(\frac{\mu\lambda(\mu+2)}{\lambda+2}\right)^{1/2} \tag{7.7a}$$

and, from (6.4),

$$\mathscr{Q}(\Lambda_2=1,\rho=2)=-\frac{1}{2}\left(\frac{\mu\lambda(\mu+2)}{\lambda+2}\right)^{1/2}\left(\frac{(\lambda+\mu+4)(2\mu+\lambda+2)}{(2\mu\lambda+2\mu^2+8\mu+3\lambda+6)}\right). \tag{7.7b}$$

Similarly

$$\mathscr{P}(\Lambda_2=0,\rho=2)=-\frac{1}{2}\left(\frac{\lambda}{3(\lambda+2)}\right)^{1/2}(2\lambda+3\mu+4) \tag{7.7c}$$

and

$$\mathscr{Q}(\Lambda_2=0,\rho=2)=-\frac{1}{2}\left(\frac{\lambda}{3(\lambda+2)}\right)^{1/2}\left(\frac{3(\mu+2)(\lambda+\mu+2)(2\mu+\lambda+2)}{(2\mu\lambda+2\mu^2+8\mu+3\lambda+6)}\right). \tag{7.7d}$$

Defining

$$\langle[\lambda\mu]\mathrm{HW};[11]00\|[\lambda\mu]\mathrm{HW}\rangle_2=y$$

we find with the help of (7.7) and the recursion formulae given by Hecht (1965):

$$\begin{aligned}
&\langle[\lambda\mu]\mathrm{HW};[11]01\|[\lambda\mu]\mathrm{HW}\rangle_2=\left(\frac{\mu}{3(\mu+2)}\right)^{1/2}\frac{\lambda+\mu+4}{\lambda+\mu+2}y\\
&\left\langle[\lambda\mu]2\lambda+\mu-3\,\frac{\mu-1}{2};[11]3\tfrac{1}{2}\|[\lambda\mu]\mathrm{HW}\right\rangle_2=-\left(\frac{2\mu(\lambda+\mu+1)}{3(\mu+1)}\right)^{1/2}\frac{1}{\lambda+\mu+2}y\\
&\left\langle[\lambda\mu]2\lambda+\mu-3\,\frac{\mu+1}{2};[11]3\tfrac{1}{2}\|[\lambda\mu]\mathrm{HW}\right\rangle_2\\
&\qquad=-\left(\frac{1}{6\lambda(\mu+1)(\mu+2)}\right)^{1/2}\left(\frac{2(2\mu\lambda+2\mu^2+8\mu+3\lambda+6)}{\lambda+\mu+2}\right)y.
\end{aligned} \tag{7.8}$$

One can already verify the orthogonality of the two coupling from (7.2) and (7.8).

We obtain from (6.7)

$$\begin{aligned}
&U([\lambda\mu][10][\lambda\mu][01];[\lambda+1\mu]_{--}[11]_{-}2)=\left(\frac{2(\lambda+2)(\lambda+\mu+3)}{3(\lambda+1)(\lambda+\mu+2)}\right)^{1/2}y\\
&U([\lambda\mu][10][\lambda\mu][01];[\lambda-1\mu+1]_{--}[11]_{-}2)\\
&\qquad=\left(\frac{(\mu+1)}{6\lambda(\lambda+1)(\mu+2)}\right)^{1/2}\frac{2(\lambda+\mu+3)(\lambda+2)}{\lambda(\lambda+\mu+2)}y
\end{aligned} \tag{7.9}$$

from which we deduce

$$\langle[\lambda\mu]\|T^{(1-1)[11]}\|[\lambda\mu]\rangle_2=-\left(\frac{\lambda+2}{3\lambda}\right)^{1/2}\frac{(\lambda+\mu+3)(\lambda+2\mu+2)}{(\lambda+\mu+2)}y. \tag{7.10}$$

When (7.10) and (7.7d) are introduced in (6.4), we finally obtain

$$y^2=\frac{3\lambda(\mu+2)(\lambda+\mu+2)^2}{2(\lambda+2)(\lambda+\mu+3)(2\mu\lambda+2\mu^2+8\mu+3\lambda+6)}. \tag{7.11}$$

Note that the above example is chosen specifically to illustrate a slightly undesirable feature of the above choice of couplings. For obvious practical reasons, one would prefer to define the first [11] coupling by means of a [11] tensor whose components are the elements of the su(3) algebra itself and the second coupling by means of a complementary [11] tensor operator (Hecht 1965). This choice would be equivalent to making the replacement (3.15) in the generic structure of the basic Bargmann tensors. While this choice would be perfectly acceptable for the [11] tensors, it leads to unnecessary complications for higher rank tensors. For example, the factorisation (3.17) would not hold any more and, as a consequence, evaluation of the matrix elements (6.1) would be recursive as in Draayer and Akiyama (1973). Null spaces inclusion properties would also be similar to theirs (see also Le Blanc 1985) but would not be as practical as the ones uncovered in § 8 as will be discussed there. We therefore favour the factorisation (3.17) with the understanding that the substitution (3.15) is more convenient for the specific case of [11] SU(3) operators.

8. Null spaces inclusion properties; ordered set of tensors

We are now left with the justification of the ordering (4.14) for the basic Bargmann tensors. We will give a partial proof in this section of a conjecture that the ordering leads to the resolution of the SU(3) outer multiplicity advocated by Hecht (1965) as discussed in § 2.

We first redefine the parametrisation (3.1) for the SU(3) basis and define, with respect to the *highest* weight state, the following quantum numbers (Hecht 1965):

$$\begin{aligned} \Lambda &= \frac{\mu}{2} + \frac{(p-q)}{2} \\ \varepsilon &= 2\lambda + \mu - 3(p+q) \end{aligned} \tag{8.1}$$

where $m_{12} = \mu + p$, $m_{22} = q$. The ε quantum number denotes the eigenvalues of the SU(3) weight operator

$$Q_0 = 2C_{33} - C_{11} - C_{22} \tag{8.2}$$

and is therefore an additive weight. Λ is equivalent to the usual angular momentum label J.

Now, the set of allowed matrix elements (when reduced with respect to the $\mathcal{U}(2)\otimes\mathrm{U}(2)$ subgroup)

$$\langle [\lambda_3\mu_3]\mathrm{HW}|\, T^{(\gamma_1\gamma_2)[\lambda_2\mu_2]}_{\mathrm{hw}} \| [\lambda_1\mu_1]\varepsilon_1\Lambda_1\rangle$$

can be ordered by *increasing* values of

$$\Lambda_1(\rho') = \frac{\mu_1}{2} + \frac{(p_1(\rho') - q_1(\rho'))}{2} \tag{8.3a}$$

with

$$\begin{aligned} p_1(\rho') + q_1(\rho') &= \tfrac{1}{3}(2\lambda_1 + \mu_1 + 2\lambda_2 + \mu_2 - 2\lambda_3 - \mu_3) \\ p_1(\rho') &= p_{1_{\min}} + (\rho' - 1) \\ q_1(\rho') &= q_{1_{\max}} - (\rho' - 1) \end{aligned} \tag{8.3b}$$

thus by *decreasing* values of $q_1(\rho')$. Similarly, we order the multiplicity set of tensors by *increasing*

$$J_2(\rho) \equiv \frac{(\gamma_1(\rho) - \gamma_2(\rho))}{2} = \frac{\lambda_2 - j(\rho) + k(\rho)}{2}$$

which is a *decreasing* function of $j(\rho)$ (see equation (4.14)).

We now consider the entries of the matrix M defined by

$$\langle[\lambda_3\mu_3]\mathrm{HW}|T^{(\gamma_1(\rho)\gamma_2(\rho))}{}^{[\lambda_2\mu_2]}_{\mathrm{hw}}|[\lambda_1\mu_1]\varepsilon_1\Lambda_1(\rho')\rangle \equiv M_{\rho\rho'}. \tag{8.4}$$

Because the tensor is of *highest* weight, the rightmost term in (3.17b) is $(T^{(-1-1)}_{-1}{}^{[10]}_{20})^j$ which will map the ket $|[\lambda_1\mu_1]\varepsilon_1\Lambda_1\rangle$ to

$$(T^{(-1-1)}_{-1}{}^{[10]}_{20})^j : |[\lambda_1\mu_1]\varepsilon_1\Lambda_1\rangle \to [\lambda_1'\mu_1']\varepsilon_1'\Lambda_1'\rangle = |[\lambda_1\mu_1-j]\varepsilon_1+2j, \Lambda_1\rangle. \tag{8.5}$$

From (8.1), we deduce

$$\begin{aligned} p_1'+q_1' &= p_1+q_1-j \\ p_1'-q_1' &= p_1-q_1+j \end{aligned} \tag{8.6}$$

or

$$\begin{aligned} p_1' &= p_1 \\ q_1' &= q_1-j. \end{aligned} \tag{8.7}$$

But we must have $q_1' \geqslant 0$. Thus we conclude that the matrix elements (8.4) vanish for $q_1 < j$. In other words, $M_{\rho\rho'}$ has some null entries in its upper triangular part. We conjecture that all its upper triangular entries are null.

We now restrict our attention to the special case of the coupling of a $[\lambda\mu]$ SU(3) representation to itself by a self-conjugate tensor $[\sigma\sigma]$. 'This is less of a special case than it appears to be at first glance, since, in a very real sense, knowledge of all self-conjugate operators in SU(3) would be tantamount to knowledge of all SU(3) tensor operators' (Louck *et al* 1975). The maximal multiplicity for this coupling is $l = \sigma+1$. Due to the $\lambda \leftrightarrow \mu$ symmetry of the null spaces of such tensors, we may restrict consideration to the case $\mu \leqslant \lambda$ without loss of generality. We easily verify the existence of the following cases for λ, μ and of the following spans for $j, k = \sigma - j$, (allowed Bargmann tensors) and for $q_1, p_1 = \sigma - q_1$, (allowed ket states) in equation (8.4).

(i) For $\lambda+\mu<\sigma$, we have $l=0$.

(ii) For $\lambda<\sigma$, $\lambda+\mu=\sigma+n$, $n\leqslant\sigma$, we have $l=n+1$ and

$$\sigma-\lambda \leqslant j \leqslant \mu \qquad \sigma-\lambda \leqslant q_1 \leqslant \mu. \tag{8.8a}$$

(iii) For $\lambda \geqslant \sigma$, $\mu \leqslant \sigma$, we have $l=\mu+1$ and

$$0 \leqslant j \leqslant \mu \qquad 0 \leqslant q_1 \leqslant \mu. \tag{8.8b}$$

(iv) For $\lambda \geqslant \sigma$, $\mu \geqslant \sigma$, we have $l=\sigma+1$ and

$$0 \leqslant j \leqslant \sigma \qquad 0 \leqslant q_1 \leqslant \sigma. \tag{8.8c}$$

From these results and equation (8.7), we deduce that, for the case of interest here, M is a *strictly lower triangular square matrix*:

$$M \sim \begin{pmatrix} \bullet & & 0 \\ \vdots & \ddots & \\ \bullet & \dots & \bullet \end{pmatrix}. \tag{8.9}$$

The significance of such a result has already been fully discussed in § 2.

We conjecture that this result will also be verified for general couplings by non-self-conjugate tensors. For example, it is easily verified for the matrix elements

$$\langle [52]\mathrm{HW}|\, T^{(\gamma_1(\rho)\gamma_2(\rho))[21]}_{\mathrm{hw}} \| [61]\varepsilon_1\Lambda_1(\rho')\rangle.$$

Unfortunately, the proof of this conjecture is hindered by the complexity of the weight structure of SU(3) which makes the determination of the allowed ket states in equation (8.4) rather difficult. We intend to address this problem in a future work.

In any event, it should be understood that the use of a Gramm-Schmidt process using the above ordering for the Bargmann tensors and the ket states in (8.4) will allow the Wigner coefficients (2.18a) to share property (8.9) of the matrix M, i.e. the coefficients will be such that

$$\langle [\lambda_1\mu_1]\varepsilon\Lambda_1(\rho');\, [\lambda_2\mu_2]\mathrm{HW} \| [\lambda_3\mu_3]\mathrm{HW}\rangle_\rho = 0 \qquad \text{for } q_1(\rho') < j(\rho). \tag{8.10}$$

It should be noted that property (8.10) is most valuable for the following reason. In practical applications, one knows beforehand the Wigner coefficients and seeks to determine the values of the reduced matrix elements of a physical operator. Matrix elements similar to (8.4) are the most easily computed and due to the triangular structure of the coefficients (8.10), the determination of the reduced matrix elements would be straightforward, the system of linear equations to be solved being already in triangular form.

We now have to examine if property (iv) of Biedenharn's abstract basic tensors (§ 2) applies to the Bargmann tensors (3.17). To answer this point, we will look at the specific example of the coupling of a unirrep $[\lambda\mu]$ to itself by the self-conjugate tensor [33], an example that we quote from Biedenharn *et al* (1985). This coupling can have a multiplicity l of up to four, i.e. $0 \leqslant l \leqslant 4$. Denoting by S_0 the set of all SU(3) unirreps such that $l=0$ for $[\lambda\mu] \in S_0$, S_1 the set of all SU(3) unirreps such that $l=1$ for $[\lambda\mu] \in S_1$, etc, we have from (8.8) that

$$\begin{aligned}
&S_0 = \{[00], [10], [20], [01], [02], [11]\}\\
&S_1 = \{[21], [12], ([\lambda 0];\ \lambda \geqslant 3), ([0\mu];\ \mu \geqslant 3)\}\\
&S_2 = \{[22], ([\lambda 1];\ \lambda \geqslant 3), ([1\mu];\ \mu \geqslant 3)\}\\
&S_3 = \{([\lambda 2];\ \lambda \geqslant 3), ([2\mu];\ \mu \geqslant 3)\}\\
&S_4 = \{[\lambda\mu] \notin S_n;\ 0 \leqslant n \leqslant 3\}.
\end{aligned} \tag{8.11}$$

The four possible tensors (3.17) are

$$T_k \equiv T^{(k,-k)[33]} \qquad 0 \leqslant k \leqslant 3 \tag{8.12}$$

with $j = 3-k$ and $i = k$. The limits of the O'Reilly summations are

$$\begin{aligned}
&0 \leqslant k \leqslant \min(3, \lambda+\mu)\\
&0 \leqslant j \leqslant \min(3, \mu, \lambda+\mu-k)\\
&\max(0, 3-\mu) \leqslant i \leqslant \min(2k, \lambda)
\end{aligned} \tag{8.13}$$

from which we easily identify which tensor T_k has vanishing ($\times$) or non-vanishing ($\bigcirc$) matrix elements between representations belonging to S_n. We have tabulated the result for easy reference:

		T_0	T_1	T_2	T_3
S_0	[00]	×	×	×	×
	[10]	×	×	×	×
	[20]	×	×	×	×
	[01]	×	×	×	×
	[02]	×	×	×	×
	[11]	×	×	×	×
S_1	[21]	×	×	○	×
	[12]	×	○	×	×
	[λ0] $\lambda \geqslant 3$	×	×	×	○
	[0μ] $\mu \geqslant 3$	○	×	×	×
S_2	[22]	×	○	○	×
	[λ1] $\lambda \geqslant 3$	×	×	○	○
	[1μ] $\mu \geqslant 3$	○	○	×	×
S_3	[λ2] $\lambda \geqslant 3$	×	○	○	○
	[2μ] $\mu \geqslant 3$	○	○	○	×.

(8.14)

Now for a set of tensors T'_k to be canonical in the sense of Biedenharn *et al*, it should have the corresponding table:

		T'_0	T'_1	T'_2	T'_3
S_0	[00]	×	×	×	×
	[10]	×	×	×	×
	[20]	×	×	×	×
	[01]	×	×	×	×
	[02]	×	×	×	×
	[11]	×	×	×	×
S_1	[21]	○	×	×	×
	[12]	○	×	×	×
	[λ0] $\lambda \geqslant 3$	○	×	×	×
	[0μ] $\mu \geqslant 3$	○	×	×	×
S_2	[22]	○	○	×	×
	[λ1] $\lambda \geqslant 3$	○	○	×	×
	[1μ] $\geqslant 3$	○	○	×	×
S_3	[λ2] $\lambda \geqslant 3$	○	○	○	×
	[2μ] $\mu \geqslant 3$	○	○	○	×

(8.15)

i.e. defining the null spaces $N_m = \bigcup_{n \leqslant m} S_n$, these canonical tensors would be such that

$$T'_k : [\lambda\mu] \to 0 \qquad \text{for} \qquad [\lambda\mu] \in N_m,\ m \leqslant k \tag{8.16}$$

where the null spaces N_m have the inclusion properties

$$N_0 \subset \ldots \subset N_m \subset N_{m+1} \subset \ldots \subset N_{\mathscr{L}}. \tag{8.17}$$

It is obvious from this counterexample that the Bargmann tensors (4.13) do not share property (iv) of Biedenharn's set of abstract tensors.

In view of the specific symmetry properties for the Wigner coefficients predicted by Biedenharn and collaborators to follow if property (iv) were satisfied, one might suspect that the resolution presented herein is less than optimal. Besides the noted correspondence between the O'Reilly series and the structure of the Bargmann tensors, the fact that the algorithm of § 6 is rather straightforward and that it agrees with the very simple solution provided by Hecht (1965), another very strong argument in its favour is that it represents a major economy in the number of distinct tensors required for the resolution of the multiplicity problem. Recall that Biedenharn's analysis calls for a set of U(3) $[\lambda_2\mu_2]$ tensors equal in number to the dimension of the SU(3) representation $[\lambda_2\mu_2]$. With the identification of the operator patterns with the $\mathcal{U}(2)$ tensorial properties of the Bargmann tensors, it is realised that our solution calls for a set of $\mathcal{U}(2)\otimes\mathrm{U}(3)$ tensors equal in number to the number of U(2) representations contained in the SU(3) representation, which is a much smaller number.

We believe that the results of this paper give a satisfactory answer to the problem of uncovering the significance of an operator pattern, 'the principal unsolved problem in the theory of tensor operators in the unitary groups' (Biedenharn *et al* 1972).

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